DUALITIES IN KOSZUL GRADED AS GORENSTEIN ALGEBRAS

ROBERTO MARTÍNEZ-VILLA

Abstract. The paper is dedicated to the study of certain non commutative graded AS Gorenstein algebras Λ [10] , [13], [14].

The main result of the paper is that for Koszul algebras Λ with Yoneda algebra Γ, such that both Λ and Γ are graded AS Gorenstein noetherian of finite local cohomology dimension on both sides, there are dualities of triangulated categories:

$\underline{gr}_{\Lambda}[\Omega^{-1}] \cong D^b(Qgr_{\Gamma})$ and $\underline{gr}_{\Gamma}[\Omega^{-1}] \cong D^b(Qgr_{\Lambda})$

where, and Qgr_{Γ} is the category of tails, this is: the category of finitely generated graded modules gr_{Γ} divided by the modules of finite length, and $D^b(Qgr_{\Gamma})$ the corresponding derived category and $\underline{gr}_{\Lambda}[\Omega^{-1}]$ the stabilization of the category of finetely generated graded Λ-modules, module the finetely generated projective modules.

1. Introduction

The paper is dedicated to the study of certain non commutative graded AS Gorenstein algebras Λ [10] , [13], [14] those which are noetherian of finite local cohomology dimension on both sides, and Koszul. We proved in [13] that the Yoneda algebra Γ of a Koszul graded AS Gorenstein algebra is again graded AS Gorenstein. We will assume in addition Λ and Γ are both noetherian and of finite local cohomology dimension on both sides.

For such algebras we can generalize the classical Bernstein-Gelfand-Gelfand [3] theorem, which says that there is an equivalence of triangulated categories: $\underline{gr}_{\Lambda}$ $\cong D^b(CohP_n)$, where $\underline{gr}_{\Lambda}$ is the stable category of the finitely generated graded Λ-modules over the exterior algebra in n variables and $D^b(CohP_n)$ is the derived category of bounded complexes of coherent sheaves on n-dimensional projective space.

This theorem was generalized in [15] and [16] as follows:

Let Λ be a finite dimensional Koszul algebra with noetherianYoneda algebra Γ. Then there is a duality of triangulated categories: $\underline{gr}_{\Lambda}[\Omega^{-1}] \cong D^b(Qgr_{\Gamma})$, where $\underline{gr}_{\Lambda}[\Omega^{-1}]$ is the stabilization of $\underline{gr}_{\Lambda}$ (in the sense of [Buchweitz], [2]) and Qgr_{Γ} is the category of tails, this is: the category of finitely generated graded modules gr_{Γ} divided by the modules of finite length, and $D^b(Qgr_{\Gamma})$ the corresponding derived category.

Date: October 22, 2012.
2000 *Mathematics Subject Classification.* Primary 05C38, 15A15; Secondary 05A15, 15A18.
Key words and phrases. local cohomology, Castelnovo-Mumford regularity.

The main result of the paper is that for Koszul algebras Λ with Yoneda algebra Γ, such that both Λ and Γ are graded AS Gorenstein noetherian of finite local cohomology dimension on both sides, there are dualities of triangulated categories: $\underline{gr}_{\Lambda}[\Omega^{-1}] \cong D^b(Qgr_{\Gamma})$ and $\underline{gr}_{\Gamma}[\Omega^{-1}] \cong D^b(Qgr_{\Lambda})$.

Thanks: I express my gratitude to Jun-ichi Miyachi for his criticism and some helpful suggestions.

2. Castelnovo-Mumford regularity

This section is dedicated to review the concepts and results developed by P. J∅rgensen in [8], [9] and to check they apply to the algebras considered in the paper, for completeness we reproduce his proofs here. The main result is the following:

Theorem 1. *Let Λ be a noetherian Koszul AS Gorenstein algebra of finite local cohomology dimension. Then for any finitely generated graded module M there is a truncation $M_{\geq k}$ such that $M_{\geq k}[k]$ is Koszul.*

To prove it we use the line of arguments given in [8] and [9] for connected graded algebras, checking that they easily extend to positively graded locally finite algebras A over a field $\Bbbk$. This is $A = \underset{i\geq 0}{\oplus} A_i$, where $A_0 = \Bbbk \times \Bbbk \times ... \times \Bbbk$ and for each $i \geq 0$ $\dim_{\Bbbk} A_i < \infty$.

We use the following notation: Given a complex Y of graded left Λ-modules we will denote by Y' the dual complex $Y' = Hom_{\Bbbk}(Y, \Bbbk)$.

Given graded Λ-modules Y, Z the degree zero maps will be denoted by $Hom_{Gr_{\Lambda}}(Y, Z)$, $Z[i]$ is the shift defined as $Z[i]\ _j = Z_{i+j}$ and $Hom_{\Lambda}(Y,\ Z) = \underset{i\in\mathbb{Z}}{\oplus} Hom_{Gr_{\Lambda}}(Y,\ Z[i])$.

Proposition 1. *Let A be a positively graded $\Bbbk$-algebra, A^{op} the opposite algebra and X, Y complexes, $X \in D^b(Gr_{A^{op}})$ and $Y \in D^-(Gr_A)$. Then $(X \overset{L}{\otimes}_A Y)' = RHom(Y, X')$.*

Proof. Let $F \to Y$ be a quasi-isomorphism from a complex of free modules F. Then $X \overset{L}{\otimes}_A Y \cong X \otimes_A F$ and $(X \otimes_A F)^n = \underset{p+q=n}{\oplus} X^p \otimes F^q$, where $F^q = \underset{J_q}{\oplus} A$, hence, $(X \otimes_A F)^n = \underset{p+q=n}{\oplus} X^p \otimes \underset{J_q}{\oplus} A = \underset{p+q=n}{\oplus}\ \underset{J_q}{\oplus} X^p$.

Therefore: $Hom_{\Bbbk}((X \otimes_A F)^n, \Bbbk) = Hom_{\Bbbk}(\underset{p+q=n}{\oplus}\underset{J_q}{\oplus} X^p, \Bbbk) = \underset{p+q=n}{\prod}\underset{J_q}{\prod} Hom_{\Bbbk}(X^p, \Bbbk)$ $= \underset{q}{\prod}\underset{J_q}{\prod} Hom_{\Bbbk}(X^{n-q}, \Bbbk)$.

In the other hand, $\mathrm{RHom}_A(\mathrm{Y},\mathrm{X}')^{-n} = \mathrm{Hom}^{\circ}(\mathrm{F},\mathrm{X}')^{-n} = \underset{q}{\prod}\mathrm{Hom}_A(\mathrm{F}^q,(\mathrm{X}')^{q-n})$ $= \underset{q}{\prod}\mathrm{Hom}_A(\underset{J_q}{\oplus}\mathrm{A},(\mathrm{X}')^{q-n}) = \underset{q}{\prod}\underset{J_q}{\prod}(\mathrm{X}')^{q-n} = \underset{q}{\prod}\underset{J_q}{\prod}(\mathrm{X}^{n-q})' = (\mathrm{X}\otimes_A\mathrm{F})'^{-n}$. □

Let's recall the definition of local cohomology dimension.

Definition 1. *Let $A = \underset{i\geq 0}{\oplus} A_i$ be a positively graded $\Bbbk$-algebra with graded Jacobson radical $\mathfrak{m} = \underset{i\geq 1}{\oplus} A_i$, define a left exact endo functor $\Gamma_{\mathfrak{m}} : Gr_A^+ \to Gr_A^+$ in the category of bounded above graded A-modules Gr_A^+ ,by $\Gamma_{\mathfrak{m}}(M) = \underset{\overrightarrow{k}}{\lim} Hom_A(A/A_{\geq k},$*

$M)$. Denote by $\Gamma_{\mathfrak{m}}^n(-)$, the n-th derived functor. It is clear that $\Gamma_{\mathfrak{m}}^n(M) = \varinjlim_k Ext_A^n(A/A_{\geq k},\ M)$. We say that A has finite local cohomology dimension, if there exist a non negative integer d such that for all $M \in Gr_A^+$ and $n \geq d$, $\Gamma_{\mathfrak{m}}^n(M) = 0$.

We refer to [5] IX Corollary 2.4a for the proof of the following:

Lemma 1. *Let A be a $\mathbb{k}$-algebra and I an injective A- A bimodule. The I is injective both as left and as a right A-module.*

In order to prove next proposition we need the following:

Lemma 2. *Let A be a positively graded left noetherian $\mathbb{k}$-algebra of finite local cohomology dimension on the left and $\{Z_i\}_{i\in K}$ a family of Γ_m -acyclic modules. Then $\underset{i\in K}{\oplus} Z_i$ is Γ_m -acyclic.*

Proof. Let $\{Z_i\}_{i\in K}$ be a family of Γ_m -acyclic modules, this is: each Z_i has an injective resolution:

$0 \to Z_i \to I_0^i \to I_1^i \to I_2^i \to ...I_k^i \to I_{k+1}^i \to ...$ such that $0 \to \Gamma_m(I_0^i) \to \Gamma_m(I_1^i) \to \Gamma_m(I_2^i) \to ...\Gamma_m(I_k^i) \to \Gamma_m(I_{k+1}^i) \to$

has homology zero except at degree zero. Since A is noetherian the exact sequence:

$0 \to \underset{i\in K}{\oplus}(Z_i) \to \underset{i\in K}{\oplus}(I_0^i) \to \underset{i\in K}{\oplus}(I_1^i) \to \underset{i\in K}{\oplus}(I_2^i) \to ... \underset{i\in K}{\oplus}(I_k^i) \to \underset{i\in K}{\oplus}(I_{k+1}^i) \to ...$

Is an injective resolution of $\underset{i\in K}{\oplus}(Z_i)$ and $\Gamma_m(\underset{i\in K}{\oplus}(I_k^i))=\varinjlim_s Hom_A(A/A_{\geq s}, \underset{i\in K}{\oplus}(I_k^i))$ and $A/A_{\geq s}$ finitely presented (again noetherian) $\varinjlim_s Hom_A(A/A_{\geq s}, \underset{i\in K}{\oplus}(I_k^i)) = \varinjlim_s \underset{i\in K}{\oplus} Hom_A(A/A_{\geq s}, (I_k^i)) = \underset{i\in K}{\oplus} \varinjlim Hom_A(A/A_{\geq s}, (I_k^i)) = \underset{i\in K}{\oplus} \Gamma_m(I_k^i)$.

In fact: $0 \to \Gamma_m(\underset{i\in K}{\oplus}(Z_i)) \to \Gamma_m(\underset{i\in K}{\oplus}(I_0^i)) \to \Gamma_m(\underset{i\in K}{\oplus}(I_1^i)) \to \Gamma_m(\underset{i\in K}{\oplus}(I_2^i)) \to ...\Gamma_m(\underset{i\in K}{\oplus}(I_k^i)) \to \Gamma_m \underset{i\in K}{\oplus}(I_{k+1}^i))$

is isomorphic to $.0 \to \underset{i\in K}{\oplus}\Gamma_m(Z_i) \to \underset{i\in K}{\oplus}\Gamma_m(I_0^i) \to \underset{i\in K}{\oplus}\Gamma_m((I_1^i) \to \underset{i\in K}{\oplus}\Gamma_m((I_2^i) \to ... \underset{i\in K}{\oplus}\Gamma_m(I_k^i) \to \underset{i\in K}{\oplus}\Gamma_m(I_{k+1}^i) \to$

the claim follows. □

Proposition 2. *Let A be a positively graded left noetherian $\mathbb{k}$-algebra of finite local cohomology dimension on the left. Then for any $X \in D^b(Gr_{A^e})$, $Y \in D^-(Gr_A)$, there is an isomorphism $R\Gamma_{\mathfrak{m}}(X \overset{L}{\otimes}_A Y) \cong R\Gamma_{\mathfrak{m}}(X) \overset{L}{\otimes}_A Y$.*

Proof. The complex X is in D^+, hence, it has an injective resolution with objects in Gr_{A^e} , $X \to I$ and $X \in D^b(Gr_{A^e})$ implies $\mathrm{H}^i(X) = 0$ for almost all i.

Assume $\mathrm{H}^i(X) = 0$ for $i > s$ and let $Z = Kerd_s$, where $d_s : I^s \to I^{s+1}$ is the differential. Hence, $0 \to Z \to I^s \to I^{s+1} \to I^{s+2}... \to I^{s+k} \to$ is an injective resolution of Z as $A - A$ bimodule.

Since A has finite local cohomology dimension, there exists an integer t such that $\Gamma_{\mathfrak{m}}^j(Z) = 0$ for $j > t$. If $Z' = \operatorname{Im} d_t$, $d_t : I^t \to I^{t+1}$ is the differential, then $\Gamma_{\mathfrak{m}}^j(Z') = 0$ for $j > 0$, this is Z' is $\Gamma_{\mathfrak{m}}$-acyclic.

The complex $Q : 0 \to I^0 \to I^1 \to ...I^t \to Z' \to 0$ is a complex $\Gamma_{\mathfrak{m}}$-acyclic which is quasi-isomorphic to I.

The $\Gamma_{\mathfrak{m}}$-acyclic complexes form an adapted class (See [7], [19]).

Let $L \to Y$ be a free resolution of Y. Then we have isomorphisms: $X \overset{L}{\otimes}_A Y \cong X \otimes_A L \cong Q \otimes_A L$.

The module $(Q \otimes_A L)^n$ is a direct sum of objects in the complex Q and A noetherian implies sums of injective is injective, therefore $Q \otimes_A L$ is $\Gamma_{\mathfrak{m}}$-acyclic.

It follows $R\Gamma_{\mathfrak{m}}(X \overset{L}{\otimes}_A Y \cong \Gamma_{\mathfrak{m}}(Q \otimes_A L)$.But we have isomorphisms:

$Hom_A(A/A_{\geq k}, (Q \otimes_A L)^n) = Hom_A(A/A_{\geq k}, Q^p \otimes_A \underset{J_{n-p}}{\oplus} A) = \underset{J_{n-p}}{\oplus} Hom_A(A/A_{\geq k}, Q^p) = Hom_A(A/A_{\geq k}, Q^p) \otimes_A \underset{J_{n-p}}{\oplus} A = Hom_A(A/A_{\geq k}, Q^p) \otimes_A L^{n-p}$.

Therefore: $\underset{\overrightarrow{k}}{\lim} Hom_A(A/A_{\geq k}, (Q \otimes_A L)^n) = (\underset{\overrightarrow{k}}{\lim} Hom_A(A/A_{\geq k}, Q^p)) \otimes_A L^{n-p}$. We are using the fact that A is noetherian, hence $A/A_{\geq k}$ is finitely presented.

We have proved: $\Gamma_{\mathfrak{m}}(Q \otimes_A L) \cong \Gamma_{\mathfrak{m}}(Q) \otimes_A L$, therefore: $R\Gamma_{\mathfrak{m}}(X \overset{L}{\otimes}_A Y) \cong R\Gamma_{\mathfrak{m}}(X) \overset{L}{\otimes}_A Y$. □

The proof of the following lemma was given in [8] and reproduced in [14], we will not give it here.

Proposition 3. *Let Λ be a positively graded $\mathbb{k}$-algebra such that the graded simple have projective resolutions consisting of finitely generated projective modules, $\mathfrak{m}$ the graded radical of Λ and $\mathfrak{m}^{op}$ the graded radical of Λ^{op}. Then for any integer k, $\Gamma^k_{\mathfrak{m}}(\Lambda) = \Gamma^k_{\mathfrak{m}^{op}}(\Lambda)$.*

We can prove now the following:

Proposition 4. *Let A be a positively graded locally finite noetherian $\mathbb{k}$-algebra of finite local cohomology dimension on both sides. Let X, Y be bounded complexes of finitely generated graded A-modules. Then there exists a natural isomorphism:*
$RHom_A(R\Gamma_{\mathfrak{m}}(X), Y) \cong RHom_A(X, Y)$.

Proof. Letting Y' be $Y' = Hom_{\mathbb{k}}(Y,\mathbb{k})$, there is an isomorphism $RHom_A(R\Gamma_{\mathfrak{m}}(X), Y) \cong RHom_A(R\Gamma_{\mathfrak{m}}(X), Y'')$.

By Proposition 1, $RHom_A(R\Gamma_{\mathfrak{m}^{op}}(A), Y'') \cong (Y' \overset{L}{\otimes}_A R\Gamma_{\mathfrak{m}^{op}}(A))'$.

By Proposition 2, $Y' \overset{L}{\otimes}_A R\Gamma_{\mathfrak{m}^{op}}(A) \cong R\Gamma_{\mathfrak{m}^{op}}(Y' \overset{L}{\otimes}_A A) \cong R\Gamma_{\mathfrak{m}^{op}}(Y')$.

Let F be a free resolution of Y, it consists of finitely generated A-modules. Hence Y' consists of finitely cogenerated injective A-modules, then of torsion modules, and $\Gamma_{\mathfrak{m}^{op}}(Y') \cong \Gamma_{\mathfrak{m}^{op}}(F') = F' \cong Y'$.

Therefore: $RHom_A(R\Gamma_{\mathfrak{m}^{op}}(A), Y) \cong Y'' \cong Y$.

Now, there are isomorphisms:

$RHom_A(R\Gamma_{\mathfrak{m}}(X), Y) \cong RHom_A(R\Gamma_{\mathfrak{m}}(A \overset{L}{\otimes}_A X), Y) \cong RHom_A(R\Gamma_{\mathfrak{m}}(A) \overset{L}{\otimes}_A X), Y) \cong RHom_A(X, RHom(R\Gamma_{\mathfrak{m}}(A), Y)$.

The last isomorphism is by adjunction and the previous one is by Proposition 2.

By Proposition 3, $RHom_A(R\Gamma_{\mathfrak{m}}(X), Y) \cong RHom_A(X, RHom(R\Gamma_{\mathfrak{m}^{op}}(A), Y)$.

It follows: $RHom_A(R\Gamma_{\mathfrak{m}}(X), Y) \cong RHom_A(X, Y)$. □

Next we have:

Lemma 3. *For complexes $X \in D^-(Gr_A)$, $Y \in D^+(Gr_A)$, there exists a spectral sequence $E_2^{m,n} = Ext^m_A(h^{-n}X, Y)$ converging to $Ext^{n+m}_A(X, Y)$.*

Proof. Let $Y \to J$ be an injective resolution. The complex X is of the form:
$X : .. \to X^{-m} \to ... \to X^{-k} \to X^{-k+1} \to ...X^{-\ell} \to 0.$

For each n, there is a complex: $Hom_A(X,\, J^n)$:
$0 \to Hom_A(X^{-\ell}, J^n) \to Hom_A(X^{-\ell-1}, J^n) \to Hom_A(X^{-k+1}, J^n) \to ...$
$Hom_A(X^{-m}, J^n) \to ...$

Since J^n is injective, $H^m(Hom_A(X,\, J^n)) \cong Hom_A(H^m(X),\, J^n)$.

If $M^{m,n} = Hom_A(X^{-m},\, J^n)$, then $M = (M^{m,n})$ is a complex in the third quadrant.

$$\begin{array}{ccccccc}
 & 0 & & 0 & & 0 & \\
 & \downarrow & & \downarrow & & \downarrow & \\
\longleftarrow & \mathrm{Hom}_A(\mathrm{X}^{-m},\mathrm{J}^0) & \longleftarrow ... & \mathrm{Hom}_A(\mathrm{X}^{-\ell-1},\mathrm{J}^0) & \longleftarrow & \mathrm{Hom}_A(\mathrm{X}^{-\ell},\mathrm{J}^0) & \longleftarrow 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
\longleftarrow & \mathrm{Hom}_A(\mathrm{X}^{-m},\mathrm{J}^1) & \longleftarrow ... & \mathrm{Hom}_A(\mathrm{X}^{-\ell-1},\mathrm{J}^1) & \longleftarrow & \mathrm{Hom}_A(\mathrm{X}^{-\ell},\mathrm{J}^1) & \longleftarrow 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & \vdots & & \vdots & & \vdots & \\
\longleftarrow & \mathrm{Hom}_A(\mathrm{X}^{-m},\mathrm{J}^t) & \longleftarrow ... & \mathrm{Hom}_A(\mathrm{X}^{-\ell-1},\mathrm{J}^t) & \longleftarrow & \mathrm{Hom}_A(\mathrm{X}^{-\ell},\mathrm{J}^t) & \longleftarrow 0 \\
 & \downarrow & & \downarrow & & \downarrow &
\end{array}$$

Taking first the horizontal homology, then the vertical homology, we obtain the spectral sequence $\mathrm{E}_2^{m,n}$=$\mathrm{Ext}_A^m(\mathrm{h}^{-n}\mathrm{X},\mathrm{Y})$ which converges to the homology of the total complex, which by definition, is $\mathrm{Ext}_A^{n+m}(\mathrm{X},\mathrm{Y})$ [24]. □

For the next lemma we need to assume either A is Gorenstein or it is of finite local cohomology dimension.

Lemma 4. *For $X \in D^-(Gr_A)$, there is a spectral sequence $E_2^{m,n} = Tor_{-m}^A(\Gamma^n_{\mathfrak{m}^{op}}(A), X)$ converging to $\Gamma_{\mathfrak{m}}^{m+n}(X)$.*

Proof. By definition, $\Gamma_{\mathfrak{m}}^m = h^m R\Gamma_{\mathfrak{m}}$. Let F be a free resolution of X.

Then we have a double complex $M^{m,n} = (\, R\Gamma_{\mathfrak{m}^{op}}A)^m \otimes F^n$.

The complex $R\Gamma_{\mathfrak{m}^{op}}A$ is bounded in the Gorenstein case. If A is of finite local cohomology dimension $R\Gamma_{\mathfrak{m}^{op}}A$, can be truncated to a bounded complex of $\Gamma_{\mathfrak{m}^{op}}$-acyclic modules.

Taking the second filtration, we obtain a spectral sequence $\mathrm{E}_2^{m,n}$= $\mathrm{Tor}_{-m}^A(\Gamma^n_{\mathfrak{m}^{op}}(\mathrm{A}),\mathrm{X})$ converging to the total complex of M.

We have isomorphisms $TotM \cong (R\Gamma_{\mathfrak{m}^{op}}A) \overset{L}{\otimes}_A X \cong (R\Gamma_{\mathfrak{m}}A) \overset{L}{\otimes}_A X \cong R\Gamma_{\mathfrak{m}}X$. □

Definition 2. *(Castelnovo-Mumford) A complex $X \in D(Gr_A)$ is called p-regular if $\Gamma^m_{\mathfrak{m}}(X)_{\geq p+1-m} = 0$ for all m.*

If X is p-regular but not p-1-regular, then we say it has Cohen Macaulay regularity p and write $CMregX = p$. If X is not p-regular for any p, the we say $CMregX = \infty$.

If X is p-regular for all p, this is $R\Gamma_{\mathfrak{m}}X = 0$, then $CMregX = -\infty$.

Artin and Schelter introduced in [1] a notion of a non commutative regular algebra that has been very important. We will use here a generalization of non commutative Gorenstein that extends the notion of Artin-Schelter regular. This is a variation of the definition given for connected algebras in [10].

Definition 3. *Let $\Bbbk$ be a field and Λ a locally finite positively graded $\Bbbk$-algebra. Then we say that Λ is graded Artin-Schelter Gorenstein (AS Gorenstein, for short) if the following conditions are satisfied:*

There exists a non negative integer n, called the graded injective dimension of Λ, such that:

i) For all graded simple S_i concentrated in degree zero and non negative integers $j \neq n$, $Ext^j_\Lambda(S_i, \Lambda) = 0$.

ii) We have an equality $Ext^n_\Lambda(S_i, \Lambda) = S'_i[-n_i]$, with S'_i a graded Λ^{op}-simple.

iii) For a non negative integer $k \neq n$, $Ext^k_{\Lambda^{op}}(Ext^n_\Lambda(S_i, \Lambda), \Lambda) = 0$ and $Ext^n_{\Lambda^{op}}(Ext^n_\Lambda(S_i, \Lambda), \Lambda) = S_i$.

We need to assume now A is graded AS Gorenstein noetherian of finite local cohomology dimension. Under this conditions the following was proved in [14].

Theorem 2. *Let Λ be a graded AS Gorenstein algebra of graded injective dimension n and such that all graded simple modules have projective resolutions consisting of finitely generated projective modules and assume Λ has finite local cohomology dimension. Then for any graded left module M there is a natural isomorphism: $D(\varinjlim Ext^i_\Lambda(\Lambda/\Lambda_{\geq k}, M)) = Ext^{n-i}_\Lambda(M,\ D(\Gamma^n_{\mathfrak{m}}(\Lambda))$, for $0 \leq i \leq n$.*

Let $D^b_{fg}(Gr_A)$ be the subcategory of $D^b(Gr_A)$ of all bounded complexes with finitely generated homology.

Let $X \in D^b_{fg}(Gr_A)$ and $X \to I$ an injective resolution. Since X is bounded, there is an integer t such that $H^k(X) = H^k(I) = 0$ for $k > t$.

As above, we can truncate I to obtain a complex $I_>$ consisting of $\Gamma_{\mathfrak{m}}$-acyclic modules, $I_> \cong X$ and $I_> \in D^b_{fg}(Gr_A)$.

We want to prove $R\Gamma_{\mathfrak{m}}(X)' \in D^b_{fg}(Gr_A)$.

$X : 0 \to X_{s_1} \xrightarrow{d_1} X_{s_2} \to ... X_{s_{\ell-1}} \xrightarrow{d_{\ell-1}} X_{s_\ell} \to 0.$

We apply induction on ℓ.

If $\ell = 1$, then X is concentrated in degree s_1 and X of finitely generated homology means X is finitely generated and it has a projective resolution:

$... \to P_k \to P_{k-1} \to ... P_1 \to P_0 \to X \to 0$ with each P_i finitely generated.

Dualizing with respect to the ring we obtain a complex:

$P^* : 0 \to P^*_0 \to P^*_1 \to ... P^*_k \to P^*_{k+1} \to ...$ with homology $H^i(P^*) = Ext^i_A(X, A)$. Since A^{op} is noetherian, each $Ext^i_A(X, A)$ is finitely generated.

But it was proved in Theorem ?, $Ext^i_A(X, A) \cong D((\varinjlim Ext^{n-i}_A(A/A_{\geq k}, X)) = (\Gamma^{n-i}_{\mathfrak{m}}(X))'$ and $Ext^i_A(X, A)$ finitely generated, implies $R\Gamma_{\mathfrak{m}}(X)' \in D^b_{fg}(Gr_A)$.

Let C be $C = \mathrm{Coker} d_{\ell-1} = H^\ell(X)$ and $B_\ell = \mathrm{Im}\, d_{\ell-1}$.

Then there is an exact sequence of complexes:

$$\begin{array}{ccccccccccc}
 & & 0 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 \to & & X_{s_1} & \to & X_{s_2} & \to ... & X_{s_{\ell-1}} & \xrightarrow{d_{\ell-1}} & B_\ell & \to 0 & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 \to & & X_{s_1} & \to & X_{s_2} & \to ... & X_{s_{\ell-1}} & \to & X_{s_\ell} & \to 0 & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & \to & C & \to 0 & \\
 & & & & & & & & \downarrow & & \\
 & & & & & & & & 0 & &
\end{array}$$

The complex:

$Y : 0 \to X_{s_1} \xrightarrow{d_1} X_{s_2} \to ... X_{s_{\ell-1}} \xrightarrow{d_{\ell-1}} B_\ell \to 0$ is quasi- isomorphic to the complex:

$0 \to X_{s_1} \xrightarrow{d_1} X_{s_2} \to ... X_{s_{\ell-2}} \xrightarrow{d_{\ell-2}} Z_{s_{\ell-1}} \to 0$ with $Z_{s_{\ell-1}} = Kerd_{\ell-1}$.

By induction hypothesis $R\Gamma_{\mathfrak{m}}(Y)' \in D^b_{fg}(Gr_A)$.

We have a triangle $Y \to X \to C \to Y[1]$ which induces a triangle:

$R\Gamma_{\mathfrak{m}}(Y) \to R\Gamma_{\mathfrak{m}}(X) \to R\Gamma_{\mathfrak{m}}(C) \to R\Gamma_{\mathfrak{m}}(Y)[1]$

By the long homology sequence, there is an exact sequence:

$\Gamma^{j-1}_{\mathfrak{m}}(C) \to \Gamma^j_{\mathfrak{m}}(Y) \to \Gamma^j_{\mathfrak{m}}(X) \to \Gamma^j_{\mathfrak{m}}(C) \to \Gamma^{j+1}_{\mathfrak{m}}(Y)$

Dualizing with respect to $\Bbbk$, there is an exact sequence:

$(\Gamma^j_{\mathfrak{m}}(C))' \to (\Gamma^j_{\mathfrak{m}}(X))' \to (\Gamma^j_{\mathfrak{m}}(Y))'$.

Using A is noetherian and induction, it follows $(\Gamma^j_{\mathfrak{m}}(X))'$ is finitely generated.

Since for any complex Z and any i there is an isomorphism $H^i(Z)' \cong H^i(Z')$.

It follows $R\Gamma_{\mathfrak{m}}(X)' \in D^b_{fg}(Gr_A)$.

Therefore $R\Gamma_{\mathfrak{m}}(X)$ is a complex with finitely cogenerated homology and each $\Gamma^j_{\mathfrak{m}}(X)$ is finitely cogenerated hence $CMregX \neq \infty$ and $CMregX \neq -\infty$.

In the graded AS Gorenstein case, there is an integer n such that $\Gamma^j_{\mathfrak{m}}(A) = \Gamma^j_{\mathfrak{m}^{op}}(A) = 0$ for $j \neq n$. According to [14], $I'_n = \Gamma^n_{\mathfrak{m}}(A) = \Gamma^n_{\mathfrak{m}^{op}}(A) = J'_n$, where $I'_n = \oplus D(P^*_j)[-n_{\sigma(j)}]$ and $J'_n = \oplus D(P_j)[-n_{\tau(j)}]$.

Since σ and τ are permutations, I'_n is cogenerated as left module in the same degrees as J'_n is cogenerated as right module and $CMreg({}_AA) = CMreg(A_A)$.

Definition 4. *(Ext-regularity) The complex $X \in D(Gr_A)$ is r-Ext-regular if $Ext^m_A(X, A_0)_{\leq -r-1-m} = 0$ for all m.*

If X is r-Ext-regular and is not (r-1)-Ext-regular we say $Ext\text{-}regular(X) = r$. If X is not r-Ext-regular for any r, then $Ext\text{-}regular(X) = \infty$ and if for all r the complex X is r-Ext-regular, this is $Ext_A(X, A_0) = 0$, then $Ext\text{-}regular(X) = -\infty$.

In [15] we gave the following definition.

Definition 5. *A complex of graded modules over a graded algebra is subdiagonal if for each i the ith module is generated in degrees at least i, provided is not zero.*

We will make use of the following:

Lemma 5. *Let A be a locally finite graded noetherian algebra over a field $\Bbbk$ and X a complex in $D^-_{fg}(Gr_A)$. Then X has a projective resolution $P \to X$ consisting of finitely generated graded projective modules such that P is subdiagonal.*

Proof. Since X has a graded projective resolution P we may consider P instead of X and prove that $P = P' \oplus P''$ where P' is a subdiagonal complex of finitely generated projective graded modules and $H^i(P'') = 0$ for all i.

$P : ... \to P_{n+1} \to P_n \to P_{n-1} \to ...P_1 \to P_0 \to 0$

There is an exact sequence: $0 \to B_1 \to P_0 \to C \to 0$ with $H^0(P) = C$ finitely generated.

Since C has a finitely generated projective cover P'_0, there is an exact commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 \to & & B_1' & \to & P_0' & \to & C & \to 0 & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 \to & & B_1 & \to & P_0 & \to & C & \to 0 & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 \to & & P_0'' & \to & P_0'' & \to & 0 & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Hence $B_1 \cong B_1' \oplus P_0''$ and B_1' has a finitely generated projective cover P_1' and there is an exact sequence: $0 \to Z_1' \to P_1' \to B_1' \to 0$.

We have an exact commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 \to & & Z_1' & \to & P_1' \oplus P_0'' & \to & B_1' \oplus P_0'' & \to 0 & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 \to & & Z_1 & \to & P_1 & \to & B_1' \oplus P_0'' & \to 0 & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 \to & & P_1'' & \to & P_1'' & \to & 0 & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Therefore: P is isomorphic to the complex:

$... \to P_n \to P_{n-1} \to ...P_2 \overset{d_2}{\to} P_1' \oplus P_0'' \oplus P_1'' \overset{d_1}{\to} P_0' \oplus P_0'' \to 0$

with $\operatorname{Im} d_2 \subseteq Z_1' \oplus P_1''$.

It follows P decomposes as $P = P' \oplus P''$ with:

$P' : .. \to P_n \to P_{n-1} \to ...P_2 \overset{d_2}{\to} P_1' \oplus P_1'' \overset{d_1}{\to} P_0' \to 0$

$P'' : 0 \to P_0'' \to P_0'' \to 0$

The projective P_0' is finitely generated.

Assume now $P = P' \oplus P''$, where $H^i(P'') = 0$ for all i and

$P' : . \to P_{n+1} \to P_n \to P_{n-1} \to ...P_1 \to P_0 \to 0$ with P_i finitely generated for $0 \leq i \leq n-2$.

Hence $B_{n-2} = \operatorname{Im} d_{n-1}$ is finitely generated, therefore it has finitely generated projective cover P_{n-1}' and as before, there is a commutative exact diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 \to & & Z_{n-1}' & \to & P_{n-1}' & \to & B_{n-2} & \to 0 & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 \to & & Z_{n-1} & \to & P_{n-1} & \to & B_{n-2} & \to 0 & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 \to & & P_{n-1}'' & \to & P_{n-1}'' & \to & 0 & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Therefore: $Z_{n-1} \cong Z_{n-1}' \oplus P_{n-1}''$.

Letting B_{n-1}be the image of $d_n$and H_{n-1} the homology $H^{n-1}(P)$, which we assume finitely generated, there is an exact sequence: $0 \to B_{n-1} \to Z_{n-1}' \oplus P_{n-1}'' \to H_{n-1} \to 0$ and an induced commutative, exact diagram:

$$\begin{array}{ccccccc}
 & 0 & & 0 & & 0 & \\
 & \downarrow & & \downarrow & & \downarrow & \\
0\to & \overline{B}_{n-1} & \to & Z'_{n-1} & \to & H'_{n-1} & \to 0\\
 & \downarrow & & \downarrow & & \downarrow & \\
0\to & B_{n-1} & \to & Z'_{n-1}\oplus P''_{n-1} & \to & H_{n-1} & \to 0\\
 & \downarrow & & \downarrow & & \downarrow & \\
0\to & B''_{n-1} & \to & P''_{n-1} & \to & H''_{n-1} & \to 0\\
 & \downarrow & & \downarrow & & \downarrow & \\
 & 0 & & 0 & & 0 &
\end{array}$$

with $\overline{B}_{n-1} = B_{n-1} \cap Z'_{n-1}$ and $H''_{n-1}$is finitely generated.

Therefore: the exact sequence: $0 \to B''_{n-1} :\to P''_{n-1} \to H''_{n-1} \to 0$ is isomorphic to the direct sum of the exact sequences:

$0 \to L_{n-1} :\to Q''_{n-1} \to H''_{n-1} \to 0$ and $0 \to Q'_{n-1} \to Q'_{n-1} \to 0 \to 0$, with Q''_{n-1} the projective cover of H''_{n-1}, hence finitely generated. Then $B''_{n-1} \cong L_{n-1} \oplus Q'_{n-1}$.

There is a commutative exact diagram:

$$\begin{array}{ccccccc}
 & 0 & & 0 & & 0 & \\
 & \downarrow & & \downarrow & & \downarrow & \\
0\to & \overline{B}_{n-1} & \to & B'_{n-1} & \to & L_{n-1} & \to 0\\
 & \downarrow & & \downarrow & & \downarrow & \\
0\to & \overline{B}_{n-1} & \to & B_{n-1} & \to & L_{n-1}\oplus Q'_{n-1} & \to 0\\
 & \downarrow & & \downarrow & & \downarrow & \\
 & 0 & \to & Q'_{n-1} & \to & Q'_{n-1} & \to 0\\
 & & & \downarrow & & \downarrow & \\
 & & & 0 & & 0 &
\end{array}$$

where $\overline{B}_{n-1}$and L_{n-1} are finitely generated. It follows $B_{n-1} \cong B'_{n-1} \oplus Q'_{n-1}$ with B'_{n-1}finitely generated.

We have an exact sequence: $0 \to B'_{n-1} \oplus Q'_{n-1} \to P'_{n-1} \oplus Q'_{n-1} \oplus Q''_{n-1} \to P_{n-2}$.

Taking the projective cover of B'_{n-1} we obtain an exact sequence: $0 \to Z'_n \to P'_n \to B'_{n-1} \to 0$. Therefore: $0 \to Z'_n \to P'_n \oplus Q'_{n-1} \to B'_{n-1} \oplus Q'_{n-1} \to 0$ is exact.

As above, P_n decomposes $P'_n \oplus Q'_{n-1} \oplus P''_n$.

We have proved that P decomposes in the direct sum of the complexes:

$. \to P_{n+1} \to P'_n \oplus P''_n \to P'_{n-1} \oplus Q''_{n-1} \to P_{n-2} ... P_1 \to P_0 \to 0$

and $0 \to Q'_{n-1} \to Q'_{n-1} \to 0.... \to 0 \to 0$, where $P'_{n-1} \oplus Q''_{n-1}$ is finitely generated. □

With the same hypothesis as in the previous lemma, let $X \in D^b_{fg}(Gr_A)$, we can choose a projective resolution of finitely generated projective graded modules: $P \to X$ such that the differential map $d_j : P_j \to P_{j-1}$ has image contained in the radical of P_{j-1} .

Hence the complex $Hom_A(P, A_0)$:

$0 \to Hom_A(P_0, A_0) \to Hom_A(P_1, A_0) \to ...Hom_A(P_n, A_0) \to ...$ has zero differential.

It follows $Ext^k_A(X, A_0) = Hom_A(P_k, A_0) \neq 0$ and $Ext_A(X, A_0) \neq 0$.

It follows $Ext\text{-}regular(X) \neq -\infty$, but $Ext\text{-}regular(X) = \infty$ is possible.

Assume $Ext\text{-}regular(X) = r$ is finite.

There is a left decomposition of A in indecomposable summands: $A = \overset{m}{\underset{i=1}{\oplus}} Q_i$ and of each projective $P_j = \overset{n}{\underset{i=1}{\oplus}} Q_i^{(m_i)}[-n_i^j]$ with $m_i \geq 0$ and n_i^j integers.

Then $Ext^j_A(X, A_0) = Hom_A(P_j, A_0) = \overset{n}{\underset{i=1}{\oplus}} D(Q_i/rQ_i)^{(m_i)}[n^j_i]$.

Therefore: $Hom_A(P_j, A_0)_k \neq 0$ if and only if for some i, $n^j_i + k = 0$. Since the resolution is subdiagonal, $n^j_i \geq j$.

By definition $Ext^j_A(X, A_0)_{\leq -r-1-j} = 0$, this means $-r - j \leq -n^j_i$ or $r \geq n^j_i - j$, for all i and $r' = \max\{n^j_i - j\}$, exists.

Then $Ext^j_A(X, A_0)_{\leq -r'-j-1} = 0$ and $Ext^j_A(X, A_0)_{-(n^i_j - j)-j} \neq 0$.

We have proved $Ext\text{-}\ reg(X) = r = \max\{n^j_i - j\}$.

Let $P : ... \to P_{n+1} \to P_n \to P_{n-1} \to ...P_1 \to P_0 \to A_0 \to 0$ and $P' : ... \to P'_{n+1} \to P'_n \to P'_{n-1} \to ...P'_1 \to P'_0 \to A_0 \to 0$ be minimal projective resolutions of A_0 as left and as right module, respectively.

Each P_j has a decomposition $P_j = \overset{m}{\underset{i=1}{\oplus}} Q_i^{(m_i)}[-n^j_i]$ and $Tor^A_n(A_0, A_0)$ is computed as the nth-homology of the complex $A_0 \otimes_A P$:

$... \to A_0 \otimes_A P_{n+1} \to A_0 \otimes_A P_n \to A_0 \otimes_A P_{n-1} \to ...A_0 \otimes_A P_1 \to A_0 \otimes_A P_0 \to 0$ and $A_0 \otimes_A P_n = A_0 \otimes_A \overset{m}{\underset{i=1}{\oplus}} Q_i^{(m_i)}[-n^n_i] = A/\mathfrak{m} \otimes_A \overset{m}{\underset{i=1}{\oplus}} Q_i^{(m_i)}[-n^n_i] \cong \overset{m}{\underset{i=1}{\oplus}} (Q_i/\mathfrak{m}Q_i)^{(m_i)}[-n^n_i]$ $\cong \overset{m}{\underset{i=1}{\oplus}} (S_i)^{(m_i)}[-n^n_i]$ and the differential of $A_0 \otimes_A P$ is zero.

Using the second resolution $Tor^A_n(A_0, A_0)$ is the nth-homology of the complex $P' \otimes_A A_0$:

$... \to P'_{n+1} \otimes_A A_0 \to P'_n \otimes_A A_0 \to P'_{n-1} \otimes_A A_0 \to ...P'_1 \otimes_A A_0 \to P'_0 \otimes_A A_0 \to 0$

Each P'_j has a decomposition $P'_j = \overset{m}{\underset{i=1}{\oplus}} Q_i'^{(m_i)}[-n'^j_i]$ and $P'_n \otimes_A A_0 = (\overset{m}{\underset{i=1}{\oplus}} Q_i'^{(m_i)}[-n'^j_i]) \otimes_A A_0 = \overset{m}{\underset{i=1}{\oplus}} (Q'_i/(Q'_i)\mathfrak{m}^{(m_i)}[-n'^j_i] \cong \overset{m}{\underset{i=1}{\oplus}} (S'_i)^{(m_i)}[-n'^j_i]$ and the differential of $P' \otimes_A A_0$ is zero.

It follows $n^j_i = n'^j_i$ for all i.

By the above remark, $Ext\text{-}\ reg_A A_0 = Ext\text{-}\ reg A_{0A} = Ext\text{-}\ reg A_0$.

We write this as a theorem.

Theorem 3. *Let A be a locally finite $\Bbbk$-algebra. Then $Ext\text{-}\ reg_A A_0 = Ext\text{-}reg A_{0A} = Ext\text{-}\ reg A_0$.*

We next have:

Theorem 4. *Let A be a noetherian graded AS Gorenstein algebra of finite local cohomology dimension. Given $X \in D^b_{fg}(Gr_A)$, $X \neq 0$. Then $Ext\text{-}\ reg(X) \leq CMreg(X) + Ext\text{-}\ reg A_0$.*

Proof. We proved above $CMreg(X) \neq -\infty$. If $Ext\text{-}\ reg A_0 = \infty$, then the inequality is trivially satisfied.

We may assume $Ext\text{-}\ reg A_0 = r$ is finite .

Let $P \to A_0$ be a minimal projective resolution. Changing notation,

$P : ...P^{(n+1)} \to P^{(n)} \to ...P^{(1)} \to P^{(0)} \to 0$

where $P^{(m)} = \oplus P_j^{(m)}[-\sigma_{m,j}]$ and $\sigma_{m,j} \leq r + m$.

Dualizing, we obtain an injective resolution I with $I^m = \oplus D(P_j^{(m)})[\sigma_{m,j}]$, of A_0 as right module.

Let p be $p = CMreg(X)$, $Z = R\Gamma_{\mathfrak{m}}(X)$ and denote by h^{-n} the homology. Then by definition we have:

$h^{-n}(Z)_{\geq p+1+n} = h^{-n}(R\Gamma_{\mathfrak{m}}(X))_{\geq p+1+n} = \Gamma^{-n}_{\mathfrak{m}}(X)_{\geq p+1+n} = 0$ for all n. Therefore: $(h^{-n}(Z))'_{\leq -p-1-n} = 0$.

But $Ext^m_A(h^{-n}(Z),\ A_0)$ is a subquotient of $Hom_A(\ h^{-n}(Z),\ I^m) = Hom_A(\ h^{-n}(Z),\ \oplus D(P^{(m)}_j)[\sigma_{m,j}])$=$\oplus Hom_A(\ h^{-n}(Z),\ D(P^{(m)}_j)[\sigma_{m,j}])\cong$

$\oplus Hom_{\Bbbk}(\ (P^{(m)}_j)^*\otimes h^{-n}(Z),\Bbbk)[\sigma_{m,j}] \cong \oplus Hom_{\Bbbk}(\ (e_j h^{-n}(Z),\Bbbk)[\sigma_{m,j}]$ with e_j the idempotent corresponding to $P^{(m)}_j$.

Since $(h^{-n}(Z))'_{\leq -p-1-n} = 0$, it follows $Hom_{\Bbbk}(\ (e_j h^{-n}(Z),\Bbbk)_{\leq -p-1-n} = 0$.

Observe that the truncation of a shifted module $M[k]_{\leq -t-k} = M_{\leq -t}[k]$.

Therefore: $Ext^m_A(h^{-n}(Z),\ A_0)_{\leq} - p - 1 - n - r - m = 0$.

We have a converging spectral sequence:

$E^{m,n}_2 = Ext^m_A(h^{-n}(Z),\ A_0) \Longrightarrow Ext^{m+n}_A(Z,\ A_0)$.

This means $Ext^{m+n}_A(Z,\ A_0)$ is a subquotient of $E^{m,n}_2 = Ext^m_A(h^{-n}(Z),\ A_0)$ and $Ext^m_A(h^{-n}(Z),\ A_0)_{\leq} - p - 1 - r - (n+m) = 0$ implies $Ext^q_A(Z,\ A_0)_{\leq -p-1-r-q} = 0$ for all q.

We have isomorphisms: $Ext^q_A(Z,\ A_0) = Ext^q_A(R\Gamma_{\mathfrak{m}}(X),\ A_0) = H^q(RHom(R\Gamma_{\mathfrak{m}}(X), A_0)) \cong H^q(RHom(X, A_0)) = Ext^q_A(X, A_0)$.

Therefore: $Ext^q_A(X, A_0)_{\leq -p-1-r-q} = 0$.

This implies $Ext\text{-}reg(X) \leq p + r = CMreg(X) + Ext\text{-}\ regA_0$. □

Corollary 1. *Assume the same conditions as in the theorem and Ext- $regA_0$ finite. Then for any $X \in D^b_{fg}(Gr_A)$, Ext- $reg(X)$ is finite.*

Proof. This follows from the above remark that $CMreg(X)$ is finite. □

Interchanging the roles of Ext-regular and CM-regular we obtain in the next result a similar inequality.

Theorem 5. *Let A be a noetherian AS Gorenstein algebra of finite local cohomology dimension. Given $X \in D^b_{fg}(Gr_A)$, $X \neq 0$. Then $CMreg(X) \leq Ext\text{-}reg(X) + CMregA$.*

Proof. Since we know $CMregA \neq -\infty$, the assumption $Ext\text{-}reg(X) = \infty$ gives the inequality and we can assume $Ext\text{-}reg(X) = r$ is finite.

As before, there is a projective resolution $P \to X$ of X with $P^{(m)} = \oplus P^{(m)}_j[-\sigma_{m,j}]$ and $\sigma_{m,j} \leq r + m$.

Let p be $p = CMreg_A A = CMregA_A$. Then by definition $\Gamma^n_{\mathfrak{m}^{op}}(A)_{\geq p+1-n} = 0$ for all n.

$\mathrm{Tor}^A_{-m}(\Gamma^n_{\mathfrak{m}^{op}}(A),X)$ is a subquotient of $\Gamma^n_{\mathfrak{m}^{op}}(A) \otimes_A P^{(-m)} = \oplus\Gamma^n_{\mathfrak{m}^{op}}(A) \otimes_A P^{(-m)}_j[-\sigma_{-m,j}] = \oplus\Gamma^n_{\mathfrak{m}^{op}}(A)e_j[-\sigma_{-m,j}]$ with e_j the idempotent corresponding to $P^{(-m)}_j$ and $\sigma_{-m,j} \leq r - m$.

Therefore: $\Gamma^n_{\mathfrak{m}^{op}}(A)[-\sigma_{-m,j}]_{\geq p+1-n+(r-m)} = 0$.

As above, it follows $\mathrm{Tor}^A_{-m}(\Gamma^n_{\mathfrak{m}^{op}}(A),X)_{\geq p+1-n+r-m} = 0$

The spectral sequence $E^{-m.n}_2 = \mathrm{Tor}^A_{-m}(\Gamma^n_{\mathfrak{m}^{op}}(A),X) \Longrightarrow \Gamma^{-m+n}_{\mathfrak{m}}(X)$ converges (Lemma 3).

Hence $\Gamma^{m+n}_{\mathfrak{m}}(X)$ is a subquotient of $\mathrm{Tor}^A_{-m}(\Gamma^n_{\mathfrak{m}^{op}}(A),X)$ and it follows $\Gamma^q_{\mathfrak{m}}(X)_{\geq p+1+r-q} = 0$.

We have proved $CMreg(X) \leq p + r = Ext\text{-}reg(X) + CMregA$. □

Remark 1. *The algebra A is Koszul if and only if $Ext\text{-}regA_0 = 0$.*

Corollary 2. *Assume the same conditions on A as in the theorem and in addition A Koszul and $CMregA = 0$. Then Ext- $reg(X) = CMreg(X)$.*

We have all the ingredients to prove the main theorem of the section.

Theorem 6. *Let A be a noetherian AS Gorenstein algebra of finite local cohomology dimension. Assume A Koszul and let M be a finitely generated graded A-module . Then for $s \geq CMregM$, the projective resolution of $M_{\geq s}[s]$ is linear.*

Proof. Assume $M_{\geq s}[s] \neq 0$ and let $P^{(n+1)} \to P^{(n)} \to ...P^{(1)} \to P^{(0)} \to M_{\geq s}[s] \to 0$ be the projective resolution. The module $M_{\geq s}[s]$ is generated in degree zero and $P^{(m)}$ decomposes as $P^{(m)} = \oplus P_j^{(m)}[-\sigma_{m,j}\ ]$ and $m \leq \sigma_{m,j}$.

We most prove $P^{(m)}$ does not have generators in degrees larger than m, or equivalently $Ext\text{-}\ reg(M_{\geq s}[s]) \leq 0$, which will follow from the above inequalities once we prove $CMreg(M_{\geq s}[s]) \leq 0$ or equivalently, $CMreg(M_{\geq s}) \leq s$, this is:

$\Gamma_{\mathfrak{m}}^m(M_{\geq s})_{\geq s+1-m} = 0$.

The module $L = M/M_{\geq s}$ is of finite length. By the local cohomology formula, $\varinjlim_k Ext_A^j(A/\mathfrak{m}^k, L) = D(Ext_A^{n-j}(L, D(\Gamma_{\mathfrak{m}}^n(A)))$.

Since A is graded AS Gorenstein $Ext_A^{n-j}(L, D(\Gamma_{\mathfrak{m}}^n(A)) = 0$ for $j \neq n$. It follows $\Gamma_{\mathfrak{m}}^j(M/M_{\geq s}) = \begin{cases} 0 & \text{if} \quad j \neq s \\ M/M_{\geq s} & \text{if} \quad j = s \end{cases}$

The exact sequence: $0 \to M_{\geq s} \to M \to M/M_{\geq s} \to 0$ induces a triangle $M_{\geq s} \to M \to M/M_{\geq s} \to M_{\geq s}[1]$, hence a triangle $R\Gamma_{\mathfrak{m}}(M_{\geq s}) \to R\Gamma_{\mathfrak{m}}(M) \to R\Gamma_{\mathfrak{m}}(M/M_{\geq s}) \to R\Gamma_{\mathfrak{m}}(M_{\geq s})[1]$, by the long homology sequence we obtain an exact sequence:

$\to \Gamma_{\mathfrak{m}}^{m-1}(M/M_{\geq s}) \to \Gamma_{\mathfrak{m}}^m(M_{\geq s}) \to \Gamma_{\mathfrak{m}}^m(M) \to \Gamma_{\mathfrak{m}}^m(M/M_{\geq s})$

The inequality $s \geq CMreg(M)$ implies $\Gamma_{\mathfrak{m}}^m(M)_{\geq s+1-m} = 0$ for all m. Since $M/M_{\geq s}$ has length s, $\Gamma_{\mathfrak{m}}^m(M/M_{\geq s})_{\geq s+1-m} = 0$ for all m.

It follows $\Gamma_{\mathfrak{m}}^m(M_{\geq s})_{\geq s+1-m} = 0$ for all m. □

3. Algebras AS Gorenstein and Koszul

In this section we will use the main theorem of the last section in order to extend a theorem by Bernstein-Gelfand-Gelfand, [3] which claims that for the exterior algebra in n-variables Λ there is an equivalence of triangulated categories $\underline{gr}_\Lambda \cong D^b(CohP_n)$ from the stable category of finitely generated graded modules to the category of bounded complexes of coherent sheaves on projective space P_n. The theorem was extended to finite dimensional Koszul algebras in [15],[16] see also [21]. We want to prove here a version of this theorem for AS Gorenstein algebras of finite cohomological dimension. We will show that the arguments used in [15] can be easily extended to this situation. We will assume the reader is familiar with the results of [13], [15] and [17] and the bibliography given there.

It was proved in [25] and [12] that a finite dimensional Koszul algebra Λ is selfinjective if and only if its Yoneda algebra Γ is Artin Schelter regular [1]. The following generalization was proved in [13] and [22] :

Theorem 7. *A Koszul algebra Λ is graded AS Gorenstein if and only if its Yoneda algebra Γ is graded AS Gorenstein.*

Remark 2. *Observe the following:*

i) The algebra Λ can be noetherian with non noetherian Yoneda algebra .

ii) The algebra Λ could be Gorenstein and Γ only weakly Gorentein this is: there exists an integer n such that for all Γ-modules left (right) of finite length $Ext^j_\Gamma(M\ ,\Gamma)=0$ for all $j>n$.

iii) The algebra Λ could be of finite local cohomology dimension and Γ of infinite local cohomology dimension.

However, there are Koszul algebras Λ with Yoneda algebra Γ such that both Λ and Γ are graded AS Gorenstein, noetherian (in both sides) and of finite cohomological dimension, for example if Λ is selfinjective with noetherian Yoneda algebra Γ then $\Lambda\otimes\Gamma$ is AS Gorenstein Koszul noetherian of finite local cohomology dimension on both sides with Yoneda algebra the skew tensor product (in the sense of [5] or [18]) $\Lambda\boxtimes\Gamma$which is also AS Gorenstein noetherian and of finite local cohomology dimension on both sides.

A concrete example of such algebras is Λ the exterior algebra in n variables and Γ the polynomial algebra in n variables, this example appears as the cohomology ring of an elementary abelian p-group over a field of positive characteristic $p\neq 2$. [4]

Another example is the trivial extension $\Lambda=\Bbbk Q\rhd D(\Bbbk Q)$ with Q an Euclidean diagram and Γ the preprojective algebra corresponding to Q [11].

We need the following definitions and results from [17]:

Definition 6. *Let Λ be a Koszul algebra with graded Jacobson radical $\mathfrak{m}$. A finitely generated graded Λ-module M is weakly Koszul if it has a minimal projective resolution:*
$\to P_n\overset{d_n}{\to}P_{n-1}\to ...P_1\to P_0\overset{d_0}{\to}M\to 0$ *such that* $\mathfrak{m}^{k+1}P_i\cap\ker d_i=\mathfrak{m}^k\ker d_i$.

The next result characterizing weakly Koszul modules was proved in [17].

Theorem 8. *Let Λ be a Koszul algebra with Yoneda algebra and denote by gr_Λ, the category of finitely generated graded Λ-modules, $F: gr_\Lambda\to Gr_\Gamma$ be the exact functor $F(M)=\underset{k\geq 0}{\oplus}Ext^k_\Lambda(M,\Lambda_0)$. Then M is weakly Koszul if and only if $F(M)$ is Koszul .*

As a consequence of this theorem and the results of the last section we have:

Theorem 9. *Let Λ be a Koszul algebra with Yoneda algebra Γ such that both are AS graded Gorenstein noetherian algebras of finite local cohomology dimension on both sides. Then given a finitely generated left Λ-module M there is a non negative integer k such that $\Omega^k(M)$ is weakly Koszul.*

Proof. Since Λ is Koszul AS graded Gorenstein noetherian algebras of finite local cohomology dimension on both sides, for any finitely generated graded Λ-module M there is a truncation $M_{\geq s}$ such that $M_{\geq s}[s]$ is Koszul and there is an exact sequence: $0\to M_{\geq s}\to M\to M/M_{\geq s}\to 0$ with $M/M_{\geq s}$ of finite length. Then we have an exact sequence: $F(M/M_{\geq s})\to F(M)\to F(M_{\geq s})$. Since F sends simple modules to indecomposable projective, it sends modules of finite length to finitely generated modules and $M_{\geq s}$ Koszul up to shift implies $F(M_{\geq s})$ Koszul up to shift, hence finitely generated. Since we are assuming Γ noetherian, it follows $F(M)$ is finitely generated. By Theorem 6, $F(M)$ has a truncation $F(M)_{\geq t}$ Koszul up to shift and $F(M)_{\geq t}=\underset{k\geq t}{\oplus}Ext^k_\Lambda(M,\Lambda_0)[-t]\cong\underset{k\geq 0}{\oplus}Ext^k_\Lambda(\Omega^t(M),\Lambda_0)[-t]=F(\Omega^t(M))$.

By Theorem 8, $\Omega^t(M)$ is weakly Koszul. □

Definition 7. *A complex of graded Λ-modules is linear if for each i, the ith module is generated in degree i , provided is not zero.*

Let Q be a finite quiver, $\Bbbk Q$ the path algebra graded by path length and $\Lambda = \Bbbk Q/I$ be a quotient with I a homogeneous ideal contained in $\Bbbk Q_{\geq 2}$ and Γ the Yoneda algebra of Λ, it was shown in [16] that there is a functor

$$\Phi : \ell.f.gr_\Lambda \to \mathfrak{lcp}_\Gamma^-$$

between the category of locally finite graded Λ-modules, $\ell.f.gr_\Lambda$, and the category of right bounded linear complexes of finitely generated graded projective Γ-modules $\mathfrak{lcp}_\Gamma^-$. We recall the construction of Φ.

Let $M = \{M_i\}_{i\geq n_0}$ be a finitely generated graded Λ-module and $\mu : \Lambda_1 \otimes_{\Lambda_0} M_k \to M_{k+1}$ the map of Λ_0-modules given by multiplication.

Since M_k is a finitely generated Λ_0-module, we have a homomorphism of Λ_0-modules

$$D(\mu) : D(M_{k+1}) \to D(M_k) \otimes_{\Lambda_0} D(\Lambda_1) \ ,$$

where $D(-) = Hom_{\Lambda_0}(-, \Lambda_0)$. Applying $Hom_\Lambda(-, \Lambda_0)$ to the exact sequence

$$0 \to \mathfrak{m} \to \Lambda \to \Lambda_0 \to 0$$

induces an exact sequence

$$0 \to \mathrm{Hom}_\Lambda(\Lambda_0,\Lambda_0) \to \mathrm{Hom}_\Lambda(\Lambda,\Lambda_0) \to \mathrm{Hom}_\Lambda(\mathfrak{m},\Lambda_0) \to \mathrm{Ext}^1_\Lambda(\Lambda_0,\Lambda_0) \to 0$$

the second map is an isomorphism, which implies $\mathrm{Hom}_\Lambda(\mathfrak{m},\Lambda_0)\to\mathrm{Ext}^1_\Lambda(\Lambda_0,\Lambda_0)$ is an isomorphism. Since Λ_0 is semisimple, there is an isomorphism

$$Hom_\Lambda(\mathfrak{m}, \Lambda_0) \cong Hom_\Lambda(\mathfrak{m}/\mathfrak{m}^2, \Lambda_0)$$

As a result there is an isomorphism $D(\Lambda_1) = \mathrm{Hom}_{\Lambda_0}(\Lambda_1, \Lambda_0) \cong \Gamma_1$ and we have a Λ_0-linear map $d_{k_0} : D(M_{k+1}) \to D(M_k) \otimes_{\Lambda_0} \Gamma_1$.

For any $\ell \geq 0$, using the fact $\Lambda_0 \cong \Gamma_0$ the multiplication map $\upsilon : \Gamma_1 \otimes_{\Gamma_0} \Gamma_\ell \to \Gamma_{\ell+1}$ induces a new map d_{k_ℓ}, as shown in the diagram:

$$\begin{array}{ccc} D(M_{k+1}) \otimes_{\Gamma_0} \Gamma_\ell & \to & D(M_k) \otimes_{\Gamma_0} \Gamma_1 \otimes_{\Gamma_0} \Gamma_\ell \\ & \searrow & \downarrow 1 \otimes \upsilon \\ d_{k_\ell} & & D(M_k) \otimes_{\Gamma_0} \Gamma_{\ell+1} \end{array}$$

Hence there is a map in degree zero

$$d_k : D(M_{k+1}) \otimes_{\Gamma_0} \Gamma\text{[-k-1]} \to D(M_k) \otimes_{\Gamma_0} \Gamma\text{[-k]}$$

Definition 8. *We call Φ the linearization functor.*

Proposition 5. *The sequence $\Phi(M) = \{D(M_{k+1}) \otimes_{\Gamma_0} \Gamma\text{[-k-1]},\ d_k\}$ is a right bounded linear complex of finitely generated graded projective Γ-modules.*

The following proposition was proved in [16]

Proposition 6. *The algebra $\Lambda = \Bbbk Q/I$ is quadratic if and only if $\Phi : \ell.f.gr_\Lambda \to \mathfrak{lcp}_\Gamma^-$ is a duality.*

We can say more in case $\Lambda = \Bbbk Q/I$ is a Koszul algebra.

Theorem 10. *Suppose $\Lambda = \Bbbk Q/I$ is a Koszul algebra and M a locally finite bounded above graded Λ-module. Then M is Koszul if and only if $\Phi(M)$ is exact, except at minimal degree; in that case, $\Phi(M)$ is a minimal projective resolution of the Koszul module (up to shift) $F(M) = \underset{k\geq t}{\oplus} Ext^k_\Lambda(M, \Lambda_0)$.*

3.1. **Approximations by linear complexes.** In this section we will see that the approximations by linear complexes given in [15] can be extended to the family of AS Gorenstein Koszul algebras considered above. Let Λ be a possibly infinite dimensional Koszul algebra with Yoneda algebra Γ. The category of complexes of finitely generated graded projective Γ-modules with bounded homology $K^{-b}(grP_\Gamma)$, module the homotopy relations, is equivalent to the derived category of bounded complexes $D^b_{fg}(Gr_\Gamma)$.

We proved in Lemma 4, that any complex X in $D^-_{fg}(Gr_\Gamma)$ has projective resolution $P \to X$ with P subdiagonal. Linear complexes are by definition subdiagonal.

Lemma 6. *Let M and N be complexes of graded modules over a graded algebra and $f : M \to N$ a null-homotopic chain map. If M is linear and N is diagonal, then $f = 0$.*

Corollary 3. *Any morphism in a derived category of modules whose domain is a bounded on the right linear complex of projective modules can be represented by a chain map.*

Since our interest is in Koszul algebras we need the following:

Definition 9. *A complex is said to be totally linear, if it is linear and each of its terms has a linear projective resolution.*

Observe that this notion is a generalization of a linear complex of projective modules.

Observe that, though the proposition below has been stated more generally than in [15], the proof is the same as in [15].

Proposition 7. *Let Γbe a noetherian graded ring and $M_\bullet = \{M_i,\ d_i\}_{n\geq i\geq 0}$ a bounded totally linear complex of finitely generated graded Γ-modules. Then there exists a bounded on the right linear complex of finitely generated projective graded modules $P_\bullet$ and a quasi-isomorphism $\mu : P_\bullet \to M_\bullet$ such that $\mu_i : P_i \to M_i$ is an epimorphism for each i.*

Proof. The approximation is constructed by induction. We start with the exact sequence: $0 \to B_0 \to M_0 \to H_0 \to 0$, take the projective cover $P_0 \to M_0 \to 0$ and complete a commutative exact diagram:

$$
\begin{array}{ccccccc}
 & 0 & & 0 & & & \\
 & \downarrow & & \downarrow & & & \\
0 \to & \Omega(M_0) & \to & \Omega(M_0) & \to & 0 & \\
 & \downarrow & & \downarrow & & & \\
0 \to & \Omega(H_0) & \to & P_0 & \to & H_0 & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow 1 & \\
0 \to & B_0 & \to & M_0 & \to & H_0 & \to 0 \\
 & \downarrow & & \downarrow & & & \\
 & 0 & & 0 & & &
\end{array}
$$

Taking the pull back we obtain a commutative exact diagram:

$$\begin{array}{ccccccc} & & & 0 & & 0 & \\ & & & \downarrow & & & \\ & 0 & \to & \Omega(M_0) & \to & \Omega(M_0) & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0\to & Z_1 & \to & W_1 & \to & \Omega(H_0) & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0\to & Z_1 & \to & M_1 & \to & B_0 & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ & 0 & & 0 & & 0 & \end{array}$$

Since M_1 and $\Omega(M_0)$ are both generated in degree one and have linear resolutions, the same is true for W_1.

It is clear that the complex $0 \to M_n \to ... \to M_2 \to W_1 \to P_0 \to 0$ is totally linear and quasi-isomorphic to $M_\bullet$and the quasi-isomorphism is an epimorphism in each degree.

Assume by induction we have constructed the totally linear complex: $0 \to M_n \to ... \to M_{j+1} \to W_j \to P_{j-1} \to ... \to P_0 \to 0$

together with a quasi-isomorphism μ to the complex $M_\bullet$ which is an epimorphism in each degrees k with $0 \leq k \leq j$ and the identity in degrees k for $j+1 \leq k \leq n$.

We have a commutative exact diagram:

$$\begin{array}{ccccccc} & 0 & & 0 & & & \\ & \downarrow & & \downarrow & & & \\ 0\to & \Omega(W_j) & \to & \Omega(W_j) & \to & 0 & \\ & \downarrow & & \downarrow & & \downarrow & \\ 0\to & K & \to & P_j & \to & W_j/B_j & \to 0 \\ & \downarrow & & \downarrow & & \downarrow 1 & \\ 0\to & B_j & \to & W_j & \to & W_j/B_j & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ & 0 & & 0 & & 0 & \end{array}$$

which induces by pullback the commutative exact diagram:

$$\begin{array}{ccccccc} & & & 0 & & 0 & \\ & & & \downarrow & & \downarrow & \\ & 0 & \to & \Omega(W_j) & \to & \Omega(W_j) & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0\to & Z_{j+1} & \to & W_{j+1} & \to & K & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0\to & Z_{j+1} & \to & M_{j+1} & \to & B_j & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ & 0 & & 0 & & 0 & \end{array}$$

By Verdier's lemma we have a complex: $P_\bullet^{(j)} : 0 \to M_n \to ... \to M_{j+2} \to W_{j+1} \to P_j \to ... \to P_0 \to 0$ and a quasi isomorphism $\dot{\mu} : P_\bullet^{(j)} \to M_\bullet$ which is the identity in degrees k such that $j+2 \leq k \leq n$ and an epimorphism in the remaining degrees.

We get by induction a totally linear complex: $P_\bullet^{(n-1)} : 0 \to W_n \to P_{n-1} \to P_{n-2} \to ... \to P_0 \to 0$ with P_j for $0 \leq j \leq n-1$ finitely generated graded projective modules generated in degree j. There is a quasi-isomorphism $\mu : P_\bullet^{(n-1)} \to M_\bullet$ such that in each degree the maps are epimorphisms.

As above, we obtain the commutative exact diagram:

$$
\begin{array}{ccccccc}
 & 0 & & 0 & & & \\
 & \downarrow & & \downarrow & & & \\
0 \to & \Omega(W_n) & \to & \Omega(W_n) & \to & 0 & \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \to & Z'_n & \to & P_n & \to & B_{n-1} & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \to & Z_n & \to & W_n & \to & B_{n-1} & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & 0 & & 0 & & 0 &
\end{array}
$$

Since W_n has a linear resolution $\Omega(W_n)$ has a linear resolution $P_\bullet^{(n+1)} \to \Omega(W_n)$.

It follows $P_\bullet^{(n+1)} \to P_n \to P_{n-1} \to P_{n-2} \to ... \to P_0 \to 0$ is a linear complex of finitely generated graded projective modules which is quasi-isomorphic to $M_\bullet$ and all the maps in the quasi-isomorphism are epimorphisms. □

We see next that for noetherian AS Gorenstein algebras of finite local cohomology any bounded complex can be approximated by a totally linear complex.

Proposition 8. *Let Γ be a Koszul algebra AS graded Gorenstein noetherian algebras of finite local cohomology dimension on both sides. Then given a bounded complex $M_\bullet$ of finitely generated graded Γ-modules, there exists a totally linear subcomplex $L_\bullet$ such that $M_\bullet/\ L_\bullet$ is a complex of modules of finite length.*

Proof. Let $M_\bullet$ be the complex $M_\bullet = \{M_j \mid 0 \leq j \leq n\}$. By Theorem 6, for each j there is a truncation $(M_j)_{\geq n_j}$ such that $(M_j)_{\geq n_j}[n_j]$ is Koszul. Taking $n = \{\max n_j\}$ each $(M_j)_{\geq n}[n]$ is Koszul. Define $L_\bullet = \{L_j \mid L_j = (M_j)_{\geq n+j}\}$. Then $L_\bullet$ is totally linear with $M_\bullet/L_\bullet$ a is a complex of modules of finite length. □

We have now the following:

Lemma 7. *Let Λ be a Koszul algebra AS graded Gorenstein noetherian algebras of finite local cohomology dimension on both sides with Yoneda algebra $\Gamma$and $\Phi : gr_\Lambda \to \mathfrak{lcp}_\Gamma^-$ the linearization functor. Then for any finitely generated module M the complex $\Phi(M)$ is contained in $\mathfrak{lcp}_\Gamma^{-,b}$, this is the homology $H^i(\Phi(M)) = 0$ for almost all i.*

Proof. According to Theorem 6, there is a truncation $M_{\geq s}$ which is Koszul up to shift, and the exact sequence $0 \to M_{\geq s} \to M \to M/M_{\geq s} \to 0$, which induces an exact sequence of complexes $0 \to \Phi(M/M_{\geq s}) \to \Phi(M) \to \Phi(M_{\geq s}) \to 0$ where $\Phi(M/M_{\geq s})$ is a finite complex and $\Phi(M_{\geq s})$ is exact, except at minimal degree, it follows by the long homology sequence that $H^i(\Phi(M) = 0$ for almost all i. □

We remarked above that the categories $D^b(gr_\Gamma)$ and $K^{-,b}(grP_\Gamma)$ are equivalent as triangulated categories, we have proved that the image of Φ is contained in $K^{-,b}(grP_\Gamma)$. Composing with the equivalence, we obtain a functor $\Phi' : gr_\Lambda \to D^b(gr_\Gamma)$.

Let $\mathcal{A}$ be an abelian category, a Serre subcategory $\mathcal{T}$ of $\mathcal{A}$ is a full subcategory with the property that for every short exact sequence of $\mathcal{A}$, say, $0 \to A \to B \to C \to 0$ the object B is in $\mathcal{T}$ if and only if A, $C \in \mathcal{T}$. By [6], we have a quotient abelian category $\mathcal{A}/\mathcal{T}$ and an exact functor $\pi : \mathcal{A} \to \mathcal{A}/\mathcal{T}$, which induces at the level of derived categories an exact functor: $D(\pi) : D(\mathcal{A}) \to D(\mathcal{A}/\mathcal{T})$. The following result is well known:

Lemma 8. *[20] The kernel of $D(\pi)$ is the full subcategory $\mathcal{K}$ with objects the complex with homology in $\mathcal{T}$ and $D(\pi)$ induces an equivalence of categories $D^*(\mathcal{A})$ $/\mathcal{K} \cong D^*(\mathcal{A}/\mathcal{T})$ for $* = +, -, b$.*

We apply the lemma in the following situation:

Let Γ be a noetherian Koszul algebra, gr_Γ the category of finitely generated graded Γ-modules. Let Qgr_Γ be the quotient category of gr_Γ by the Serre subcategory of the modules of finite length. Let $\pi : gr_\Gamma \to Qgr_\Gamma$ be the natural projection and $D(\pi) : D^b(gr_\Gamma) \to D^b(Qgr_\Gamma)$ the induced functor. Denote by $\mathcal{F}_\Gamma$ be the full subcategory of $D^b(gr_\Gamma)$ consisting of bounded complexes of graded Γ-modules of finite length. Then we have:

Theorem 11. *[16]The functor $D(\pi) : D^b(gr_\Gamma) \to D^b(Qgr_\Gamma)$ has kernel $\mathcal{F}_\Gamma$. It induces an equivalence of triangulated categories $\sigma : D^b(gr_\Gamma)\ /\mathcal{F}_\Gamma \to D^b(Qgr_\Gamma)$.*

Let $q : D^b(gr_\Gamma) \to D^b(gr_\Gamma)\ /\mathcal{F}_\Gamma$ be the quotient functor. Then $\sigma q = D(\pi)$. The functor $j : K^{-,b}(grP_\Gamma) \to D^b(gr_\Gamma)$ is truncation, j is an equivalence.

Let Λ be a Koszul algebra with Yoneda algebra Γ such that both are AS graded Gorenstein noetherian algebras of finite local cohomology dimension on both sides. The functor $\theta : gr_\Lambda \to D^b(Qgr_\Gamma)$ is the composition: $gr_\Lambda \overset{\Phi}{\to} \mathfrak{lcp}_\Gamma^{-,b} \overset{i}{\to} K^{-,b}(grP_\Gamma) \overset{j}{\to} D^b(gr_\Gamma) \overset{D(\pi)}{\to} D^b(Qgr_\Gamma)$, where i is just the inclusion.

$$\begin{array}{ccccccc}
\mathfrak{lcp}_\Gamma^{-,b} & \overset{i}{\to} & \mathrm{K}^{-,b}(\mathrm{grP}_\Gamma) & \overset{j}{\to} & \mathrm{D}^b(\mathrm{gr}_\Gamma) & \overset{q}{\longrightarrow} & \mathrm{D}^b(\mathrm{gr}_\Gamma)/\mathcal{F}_\Gamma \\
\Phi\uparrow & & & & \mathrm{D}(\pi)\downarrow & \sigma\swarrow & \\
\mathrm{gr}_\Lambda & & \overset{\theta}{\to} & & \mathrm{D}^b(\mathrm{Qgr}_\Gamma) & &
\end{array}$$

Now let P be a finitely generated projective graded Λ-module, $P = \oplus P_i[n_i]$, with each P_i generated in degree zero. Then $\Phi(P)$ is isomorphic in the category of complexes over gr_Γ to $\oplus\Phi(P_i)[n_i]$ and each $\Phi(P_i)$ is a projective resolution of a semisimple Γ-module. It follows θ sends any map factoring through a graded projective module to a zero map in $\mathrm{D}^b(\mathrm{Qgr}_\Gamma)$. Consequently, θ induces a functor $\underline{\theta}$:$\underline{gr}_\Gamma$ $\to\mathrm{D}^b(\mathrm{Qgr}_\Gamma)$. The functor θ sends exact sequences to exact triangles, the syzygy functor Ω :$\underline{gr}_\Lambda \to\underline{gr}_\Lambda$ is an endofunctor that makes $\underline{gr}_\Lambda$ "half" triangulated, given an exact sequence $0 \to A \overset{j}{\to} B \overset{t}{\to} C \to 0$ in gr_Λ and $p : P \to C$ the projective cover, there is an induced exact commutative diagram:

$$\begin{array}{cccccccc}
0\to & \Omega(C) & \to & P & \to & C & \to 0 \\
& w\downarrow & & \downarrow & & \downarrow 1 & \\
0\to & A & \to & B & \to & C & \to 0
\end{array}$$

We obtain a half triangle: $\Omega(C) \to A \to B \to C$ and $\underline{\theta}$ sends the half triangle into a triangle in $\mathrm{D}^b(\mathrm{Qgr}_\Gamma)$. We want to construct a triangulated category $\underline{gr}_\Lambda[\Omega^{-1}]$ such that Ω is an equivalence which acts as the shift and a functor of half triangulated categories λ :$\underline{gr}_\Lambda \to\underline{gr}_\Lambda[\Omega^{-1}]$ such that given any triangulated category D and a functor of half triangulated categories: β $\underline{gr}_\Lambda \to D$ there is a unique functor of triangulated categories $\overset{\wedge}{\beta} : \underline{gr}_\Lambda[\Omega^{-1}] \to D$ such that $\overset{\wedge}{\beta}\lambda = \beta$.

We recall the construction given by Buchweitz and reproduced in [2], [15].

Let $(\mathcal{A}, \phi)$ be a category with endofunctor, if $(\mathcal{B},\psi)$ is another pair, then a functor $F : \mathcal{A} \to \mathcal{B}$ is said a morphism of pairs if it makes the diagram

$$\begin{tikzcd} \mathcal{A} \arrow[r, "\phi"] \arrow[d, "F"] & \mathcal{A} \arrow[d, "F"] \\ \mathcal{B} \arrow[r, "\psi"] & \mathcal{B} \end{tikzcd}$$

commute, this is: the functors $F\phi$ and ψF are naturally isomorphic. If ψ happens to be an auto equivalence, we say that the morphism F inverts ϕ.Then there is a the following universal problem. Given a pair $(\mathcal{A},\ \phi)$, find a pair $(\mathcal{A}[\phi^{-1}],\rho)$ and a morphism of pairs $G : (\mathcal{A},\ \phi) \to (\mathcal{A}[\phi^{-1}],\rho)$ such that G inverts ϕ and for any morphism of pairs $F : (\mathcal{A},\ \phi) \to (\mathcal{B},\psi)$ such that F inverts ϕ, there is a unique morphism of pairs $F' : (\mathcal{A}[\phi^{-1}],\rho) \to (\mathcal{B},\psi)$ making the diagram

$$\begin{tikzcd} (\mathcal{A},\phi) \arrow[rr, "F"] \arrow[dr, "G"'] & & (\mathcal{B},\psi) \\ & (\mathcal{A}[\phi^{-1}],\rho) \arrow[ur, "F'"'] & \end{tikzcd}$$

Commute.

The objects of $\mathcal{A}[\phi^{-1}]$ are the formal symbols $\phi^{-n}M$ where M is an object of $\mathcal{A}$ and $n \geq 0$, $\phi^0 M = M$. If M, N are objects in $\mathcal{A}[\phi^{-1}]$,we define the morphisms by

$$Mor_{\mathcal{A}[\phi^{-1}]}(M,N) = \varinjlim_k Mor_{\mathcal{A}}(\phi^k M, \phi^k N)$$

where we assume $M = \phi^{-m}M'$ and $N = \phi^{-n}N'$ and $k \geq \max\{m,n\}$.(See [MM] for details)

We define the endofunctor $\rho : \mathcal{A}[\phi^{-1}] \to \mathcal{A}[\phi^{-1}]$ by setting $\rho(M) = \phi(M)$ and $\rho(\phi^{-n}M) = \phi^{-n+1}(M)$ for any M in $\mathcal{A}$ and any natural number n. If f is a morphism represented by some $f_n : \phi^n M \to \phi^n N$ and n sufficiently large, then $\rho(f)$ is represented by $\phi(f_n)$.

We obtain the morphism of pairs $G : (\mathcal{A},\ \phi) \to (\mathcal{A}[\phi^{-1}],\rho)$ having the desired properties.

We apply this construction to our pair $(\underline{gr}_\Lambda, \Omega)$ to obtain a pair $(\underline{gr}_\Lambda[\Omega^{-1}],\ \Omega)$ and a map of pairs $G : (\underline{gr}_\Lambda, \Omega) \to (\underline{gr}_\Lambda[\Omega^{-1}],\ \Omega^{-1})$

One can check as in [15] or [2] that $(\underline{gr}_\Lambda[\Omega^{-1}],\ \Omega^{-1})$ is a triangulated category and $\underline{\theta}$:$\underline{gr}_\Lambda \to D^b(\mathrm{gr}_\Gamma)$ induces an exact functor $\hat{\theta} : \underline{gr}_\Lambda[\Omega^{-1}] \to D^b(\mathrm{Qgr}_\Gamma)$ such that the triangle

$$\begin{tikzcd} \underline{gr}_\Lambda \arrow[d, "\lambda"'] \arrow[dr, "\underline{\theta}"] & \\ \underline{gr}_\Lambda[\Omega^{-1}] \arrow[r, "\hat{\theta}"] & D^b(Qgr_\Gamma) \end{tikzcd}$$

We now state the main result of the paper.

Theorem 12. *Let Λ be a Koszul algebra with Yoneda algebra Γ such that both are AS graded Gorenstein noetherian algebras of finite local cohomology dimension on both sides. Then the linearization functor*

$\hat{\theta} : \underline{gr}_\Lambda[\Omega^{-1}] \to D^b(Qgr_\Gamma)$ *is a duality of triangulated categories.*

Proof. We will only check the functor $\hat{\theta}$ is dense, for the rest of the proof we proceed as in [15].

Choose any bounded complex $B_\bullet$ of finitely generated graded Γ-modules. By Proposition 8, the complex $B_\bullet$ is isomorphic in $D^b(Qgr_\Gamma)$ to a totally linear complex, which is in turn, by Proposition 7, isomorphic to a linear complex $P_\bullet$ of finitely

generated graded projective Γ-modules with zero homology except for a finite number of indices. By Proposition 6, there is a finitely generated graded Λ-module M such that $\Phi(M) \cong P_\bullet$. Therefore : $\overset{\wedge}{\theta}(M) \cong B_\bullet$ in $D^b(Qgr_\Gamma)$. □

Corollary 4. *Let Λ be a Koszul algebra with Yoneda algebra Γ such that both are AS graded Gorenstein noetherian algebras of finite local cohomology dimension on both sides. Then the linearization functor $\overset{\wedge}{\theta}' : \underline{gr}_\Gamma[\Omega^{-1}] \to D^b(Qgr_\Lambda)$ is a duality of triangulated categories.*

Proof. It follows by symmetry. □

References

[1] Artin M., Schelter W. Graded algebras of global dimension 3, Adv. Math. 66 (1987), 171-216.

[2] Beligiannis A. The homological theory of contravariantly finite subcategories: Auslander-Buchweitz Contexts, Gorenstein Categories and (C9)-Stabilizations, Comm. in Algebra 28 (19), (2000), 4547-4596.

[3] Bernstein J., Gelfand I.M., Gelfand S.I., Algebraic vector bundles over P^n and problems of linear algebra. Finkt. Anal. Prilozh. 12, No. 3, 66-67, (1978) English transl. Funct. Anal. Appl. 12, 212-214 (1979)

[4] Carlson J.F. The Varieties and the Cohomology Ring of a Module, J. of Algebra, Vol. 85, No. 1, (1983) 104-143.

[5] Cartan H. Eilenberg S. Homological Algebra, Princeton Mathematical Series 19, Princeton University Press 1956.

[6] Gabriel P. Des Catégories abeliennes, Bull. Soc. Math. France, 90 (1962), 323-448.

[7] Gelfand S.I., Manin Yu. I., Methods of homological algebra, Springer-Verlag (1996).

[8] J∅rgensen P. Local Cohomology for Non Commutative Graded Algebras. Comm. in Algebra, 25(2), 575-591 (1997)

[9] J∅rgensen P. Linear free resolutions over non-commutative algebras. Compositio Math. 140 (2004) 1053-1058.

[10] Jørgensen, P.; Zhang, James J. Gourmet's guide to Gorensteinness. Adv. Math. 151 (2000), no. 2, 313–345.

[11] Martínez-Villa, R. Applications of Koszul algebras: the preprojective algebra. Representation theory of algebras (Cocoyoc, 1994), 487–504, CMS Conf. Proc., 18, Amer. Math. Soc., Providence, RI, 1996.

[12] Martinez-Villa, R. Graded, Selfinjective, and Koszul Algebras, J. Algebra 215, 34-72 1999

[13] Martinez-Villa, R. Koszul algebras and the Gorenstein condition. Representations of algebras (São Paulo, 1999), 135–156, Lecture Notes in Pure and Appl. Math., 224, Dekker, New York, 2002.

[14] Martinez-Villa, R. Local cohomology and non commutative Gorenstein Algebras, Preprint, Centro de Ciencias Matemáticas, UNAM (2012).

[15] Martínez-Villa, R., Martsinkovsky, A. Stable Projective Homotopy Theory of Modules, Tails, and Koszul Duality, Comm. Algebra 38 (2010), no. 10, 3941–3973.

[16] Martínez Villa, R; Saorín, M. Koszul equivalences and dualities. Pacific J. Math. 214 (2004), no. 2, 359–378.

[17] Martínez-Villa, R.; Zacharia, Dan Approximations with modules having linear resolutions. J. Algebra 266 (2003), no. 2, 671–697.

[18] Martínez-Villa, R. ; Zacharia, Dan Selfinjective Koszul algebras. Théories d'homologie, représentations et algèbres de Hopf. AMA Algebra Montp. Announc. 2003, Paper 5, 5 pp. (electronic).

[19] Miyachi, Jun-Ichi, Derived Categories with Applications to Representation of Algebras, Chiba University, June 2000.

[20] Miyachi, Jun-Ichi, Localization of triangulated categories and derived categories, J. Algebra, 141 (1991), 463-483.

[21] Mazorchuk, V. Ovsienko, S. A pairing in homology and the category of linear complexes of tilting modules for a quasi-hereditary algebra, J. Math. Kyoto Univ. 45 (2005) no. 4, 711-741.

[22] Mori, I., Rationality of the Poincare series for Koszul algebras, Journal of Algebra, V. 276, no. 2, (2004) pag. 602-624.

[23] Popescu N. Abelian categories with applications to rings and modules, Academic Press (1973).

[24] Rotman J.J. An Introduction to Homological Algebra, Second Edition Universitext, Springer, 2009.

[25] Smith P. Some finite dimensional algebras related to elliptic curves, "Rep. Theory of Algebras and Related Topics", CMS Conference Proceedings, Vol. 19, 315-348, Amer. Math. Soc. Providence, 1996.

Centro de Ciencias Matemáticas, UNAM, Morelia
E-mail address: `mvilla@matmor.unam.mx`
URL: `http://www.matmor.unam.mx`